Machine Learning Lab
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Dimensionality Reduction

Machine Learning
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Motivation

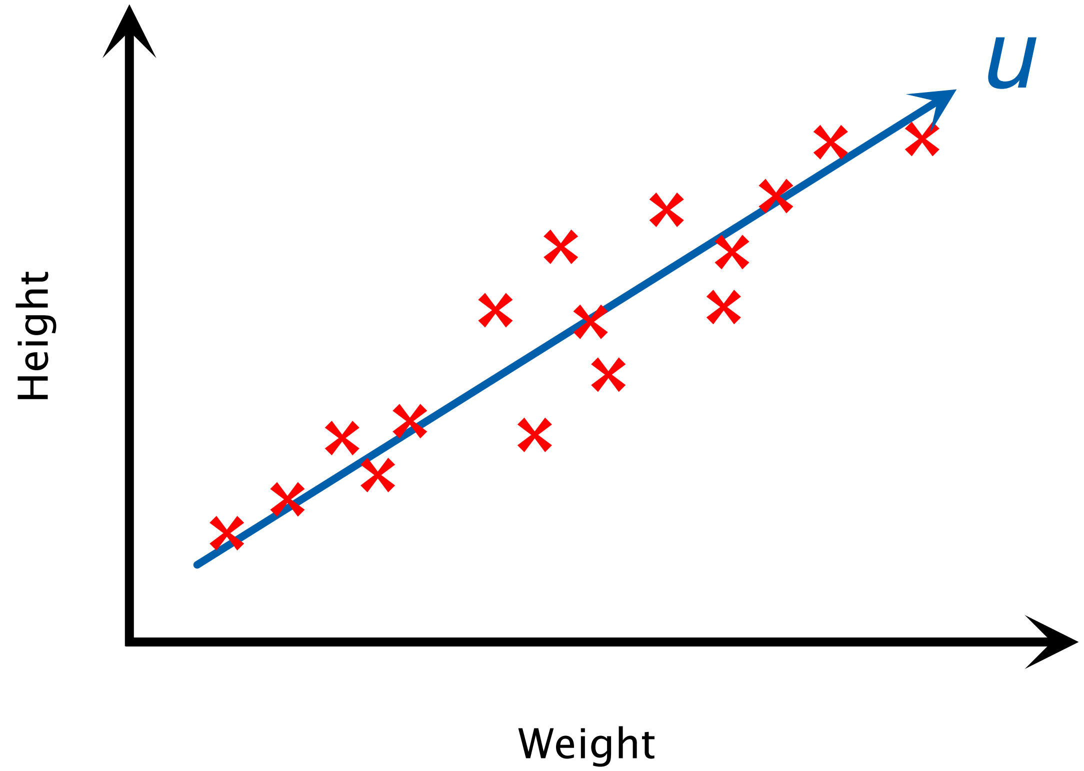
dimensionality reduction transforms a n-dimensional dataset to a k-dimensional dataset with $k < n$

- dataset compression
  - less memory storage consumption
  - machine learning algorithms run faster on low-dimensional data

- data visualization
  - high-dimensional data can be transformed to 2D or 3D for plotting

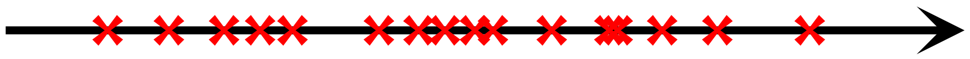
Principal Component Analysis

- most commonly used dimensionality reduction method
- projects the data on *k* orthogonal bases vectors *u* that minimize the projection error



Example:

- original 2D dataset containing features *weight* and *height*



- projection on vector *u*

PCA Algorithm

input: $x^{(1)}, x^{(2)}, \ldots, x^{(m)}$

preprocessing:

- mean normalization
  1. compute mean of each feature *j*

     $$\mu_j = \frac{1}{m}\sum_{i=1}^{m} x_j^{(i)}$$

  2. subtract the mean from data

     $$x_j^{(i)} \leftarrow x_j^{(i)} - \mu_j$$

- feature scaling

  $$x_j^{(i)} \leftarrow a_j x_j^{(i)}$$

PCA Algorithm

compute covariance matrix $\Sigma = \frac{1}{m}\sum_{i=1}^{m} x^{(i)} x^{(i)T}$

diagonalize covariance matrix (using SVD)

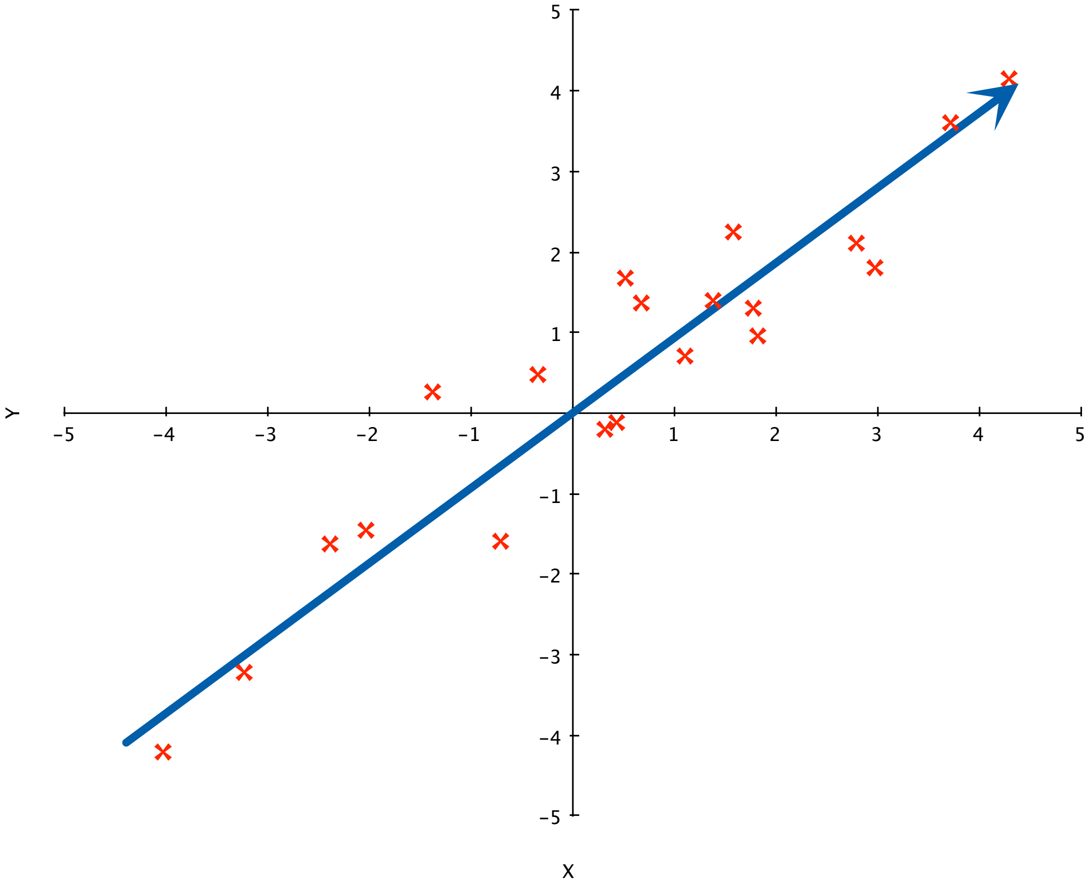
$$S = U^{-1}\Sigma U$$

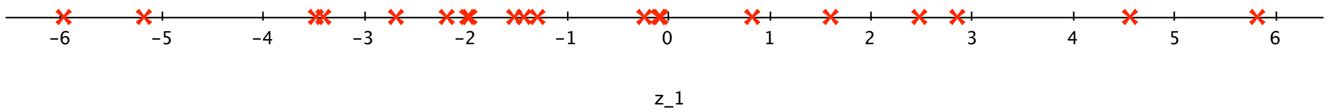
U is the matrix of Eigenvectors
S a diagonal matrix containing the Eigenvalues

dimensionality reduction from *n* to *k* dimensions:
project the data onto the Eigenvectors corresponding to the *k* largest Eigenvalues

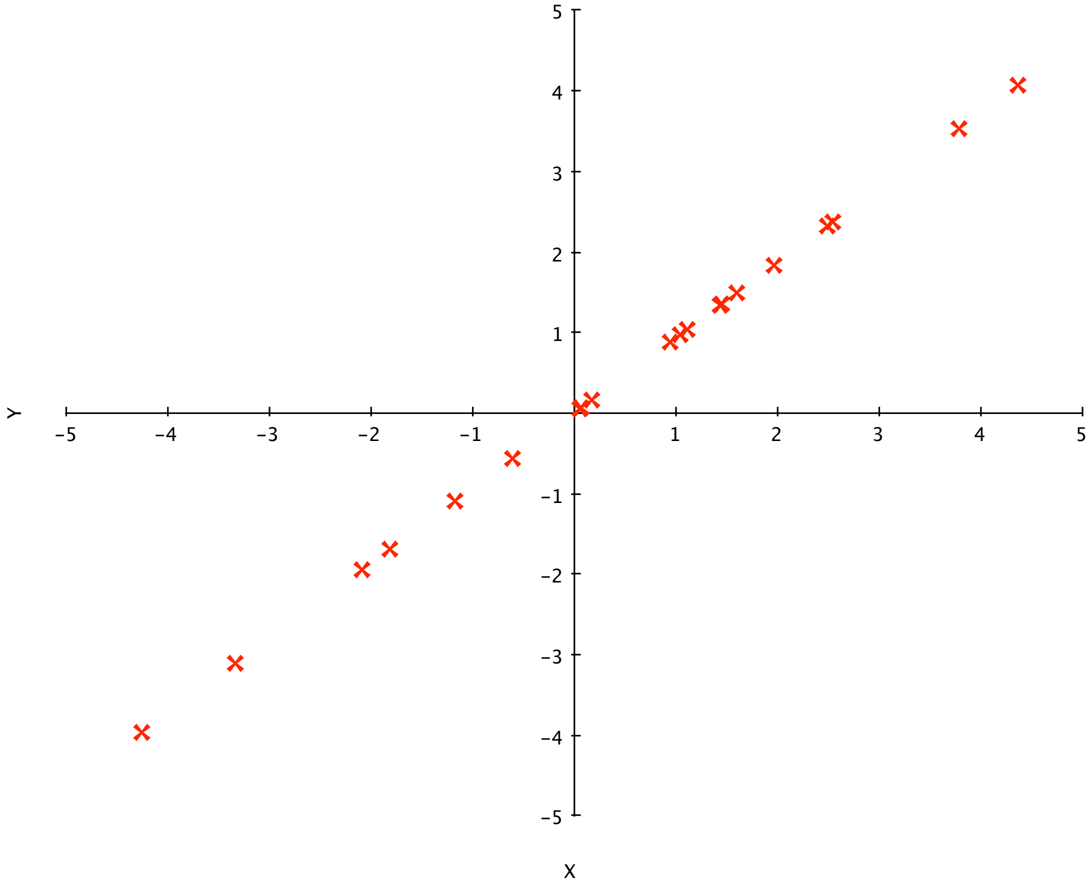
$$z^{(i)} = U_{reduce}^{T} x^{(i)}$$

Reconstruction





$$z^{(i)} = U_{reduce}^{T} x^{(i)}$$



$$x_{approx} = U_{reduce} * z^{(i)}$$

the reconstruction of compressed data points is an approximation of the original data

Choosing *k*

average squared projection error: $$\frac{1}{m}\sum_{i=1}^{m}\left\|x^{(i)} - x_{approx}^{(i)}\right\|^2$$

total variation in the data: $$\frac{1}{m}\sum_{i=1}^{m}\left\|x^{(i)}\right\|^2$$

to retain 99% of the variance, choose k to be the smallest value, such that

$$\frac{\frac{1}{m}\sum_{i=1}^{m}\left\|x^{(i)} - x_{approx}^{(i)}\right\|^2}{\frac{1}{m}\sum_{i=1}^{m}\left\|x^{(i)}\right\|^2} = 1 - \frac{\sum_{i=1}^{k} S_{ii}}{\sum_{i=1}^{n} S_{ii}} \leq 0.01$$

$$\frac{\sum_{i=1}^{k} S_{ii}}{\sum_{i=1}^{n} S_{ii}} \geq 0.99$$

Example using Real-world Data

http://archive.ics.uci.edu/ml/

- offers 223 datasets
- datasets can be used for the evaluation of ML methods
- results can be compared to those of other researchers

Iris Data Set

Download: Data Folder, Data Set Description

Abstract: Famous database; from Fisher, 1936

Data Set Characteristics:	**Multivariate**	**Number of Instances:**	**150**	**Area:**	**Life**
Attribute Characteristics:	**Real**	**Number of Attributes:**	**4**	**Date Donated**	**1988-07-01**
Associated Tasks:	**Classification**	**Missing Values?**	**No**	**Number of Web Hits:**	**348488**

Source:

Creator: R.A. Fisher

Donor: Michael Marshall (MARSHALL%PLU **'@'** io.arc.nasa.gov)

Data Set Information:

This is perhaps the best known database to be found in the pattern recognition literature. Fisher's paper is a classic in the field and is referenced frequently to this day. (See Duda & Hart, for example.) The data set contains 3 classes of 50 instances each, where each class refers to a type of iris plant. One class is linearly separable from the other 2; the latter are NOT linearly separable from each other.

Predicted attribute: class of iris plant.

This is an exceedingly simple domain.

This data differs from the data presented in Fishers article (identified by Steve Chadwick, spchadwick **'@'** espeedaz.net). The 35th sample should be: 4.9,3.1,1.5,0.2,"Iris-setosa" where the error is in the fourth feature. The 38th sample: 4.9,3.6,1.4,0.1,"Iris-setosa" where the errors are in the second and third features.

Attribute Information:

1. sepal length in cm
2. sepal width in cm
3. petal length in cm
4. petal width in cm
5. class: Iris Setosa, Iris Versicolour, Iris Virginica

PCA on the Iris dataset

given: data matrix X

preprocessing:
- mean normalization
- feature scaling

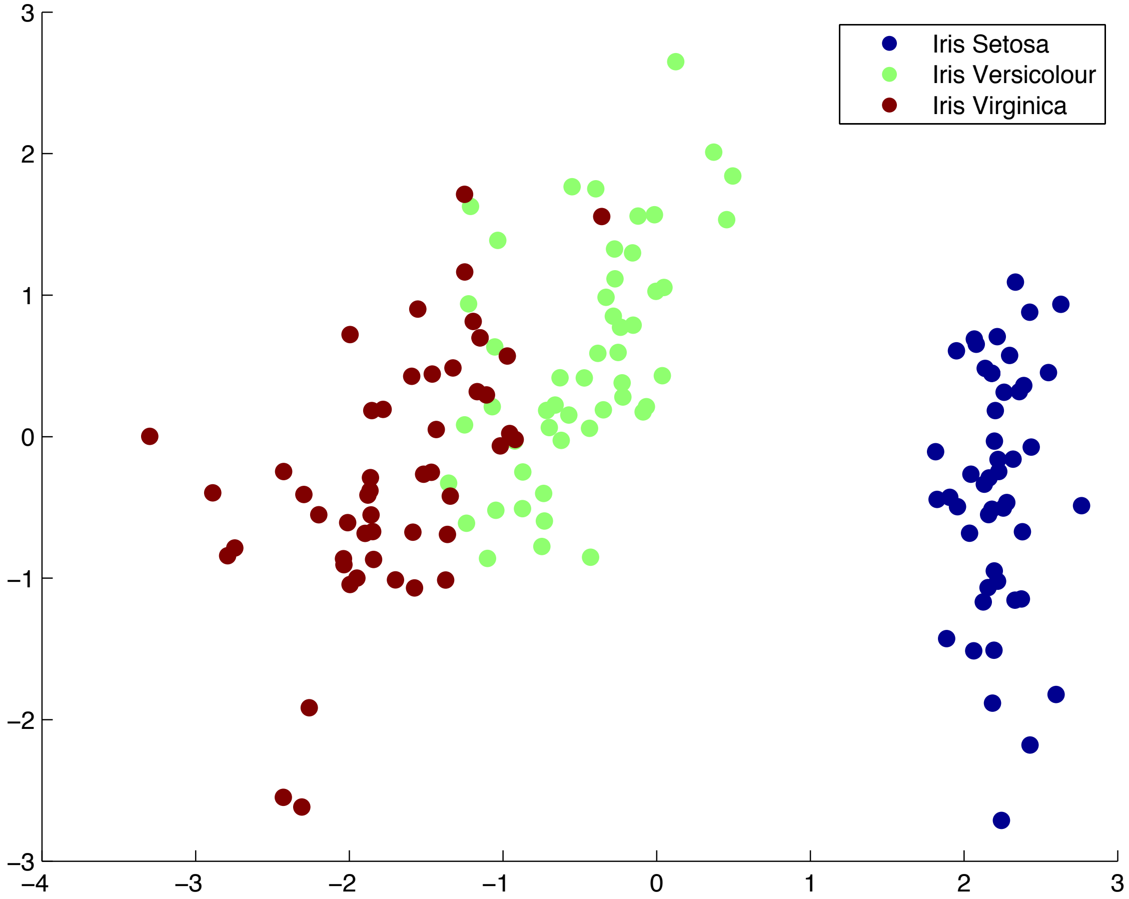
compute covariance matrix:

$$\Sigma = \frac{1}{m}\sum_{i=1}^{m} x^{(i)} x^{(i)T}$$

compute eigenvectors and eigenvalues:

$$U = \begin{pmatrix} -0.5224 & -0.3723 & 0.7210 & 0.2620 \\ 0.2634 & -0.9256 & -0.2420 & -0.1241 \\ -0.5813 & -0.0211 & -0.1409 & -0.8012 \\ -0.5656 & -0.0654 & -0.6338 & 0.5235 \end{pmatrix}$$

$$S = \begin{pmatrix} 2.8914 & 0 & 0 & 0 \\ 0 & 0.9151 & 0 & 0 \\ 0 & 0 & 0.1464 & 0 \\ 0 & 0 & 0 & 0.0205 \end{pmatrix}$$

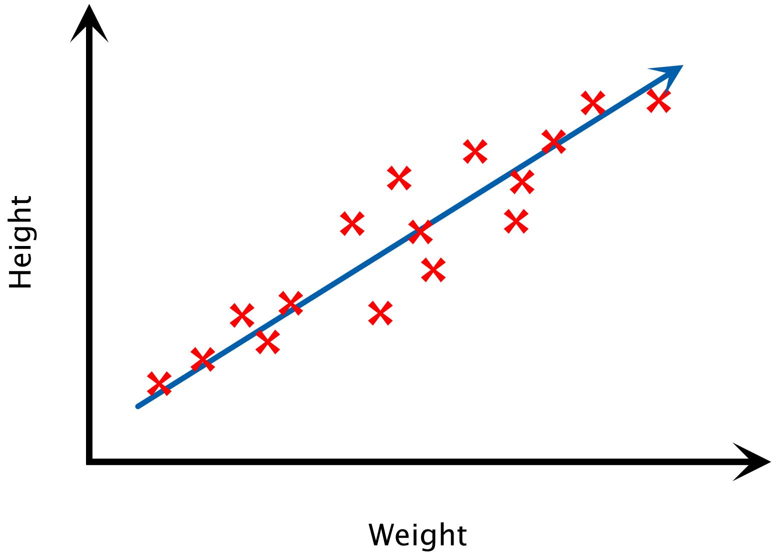


reduce U to *k* components

$$z^{(i)} = U_{reduce}^{T} x^{(i)}$$

Final Remarks

- PCA can only realize linear transformations
- there exist nonlinear extensions (Kernel PCA)
- PCA-transformed data is uncorrelated
- PCA assumes that most of the information is contained in the direction with the highest variance
- PCA is often used to reduce the noise in a signal
- PCA is an unsupervised method - when used as a preprocessing step for supervised learning the performance can drop significantly


Height
Weight